Another Look at the Birnbaum Saunders Distribution

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Abstract

The two-parameter Birnbaum-Saunders distribution was derived in 1969 as a lifetime model for a specimen subjected to cyclic patterns of stresses and strains, and the ultimate failure of the specimen is assumed to be due to the growth of a dominant crack in the material. The derivation of this model is reviewed, and it is argued that the assumptions are very restrictive and not valid for many experimental situations. Here, we consider relaxing the assumption of independent crack extensions by considering the sequence of crack extensions as a long memory process. When this is incorporated into the derivation, the resulting model is a new three-parameter generalization to the original Birnbaum-Saunders distribution. Properties of this model are discussed.

1. Introduction

Birnbaum and Saunders (1969a) derive a new family of lifetime distribution. This distribution is founded on modeling the failure of a specimen subjected to a cyclic pattern stresses and strains, and the ultimate failure is due to the growth of a dominant crack in the material. At each increment of load, this dominant crack extends by a non-negative amount, and this is assumed to be random due to variations in the material, variations in the load stresses, and so forth. Since the derivation of their model is an important aspect for this article, we will summarize their methods below.

Consider a specimen that is subjected to a series of cyclic, continuous loadings. Here, it is implied that at any given moment in time, "load" is a function that represents the amount of stress put upon the specimen at this time. The fatigue failure of the specimen is due to the growth and ultimate extension of a dominant crack within the specimen. The Birnbaum-Saunders distribution arises from the following steps:

A cycle is defined as m oscillations, and each application of the i^{th} oscillation in a cycle results in a random crack extension X_i. The distribution for this random variable depends only on the actual crack extensions caused by prior load oscillations in this cycle only.

The crack extension due to the j^{th} cycle is given by $Y_j = \sum_{i=1}^{m} X_i$, and this is a random variable with some mean μ and variance σ^2 for all $j = 1, 2, \ldots$. The total crack extension after n cycles would be given by $W_n = \sum_{j=1}^{n} Y_j$ with distribution function $H_n(w) = P(W_n \le w)$, for $n = 1, 2, \ldots$.

Let C represent the *number of cycles* until failure, where failure is defined as the dominant crack length exceeding some critical length ω. The distribution function for C is given by $P(C \le n) = 1 - H_n(\omega)$.

The Y_j are assumed to be independent and identically distributed random variables, so the distribution of W_n can be well approximated using the Central Limit Theorem (CLT). Namely,

$$P(C \le n) = P\left[\sum_{j=1}^{n} \frac{Y_j - \mu}{\sigma\sqrt{n}} > \frac{\omega - n\mu}{\sigma\sqrt{n}}\right] \approx \Phi\left[\frac{\mu\sqrt{n}}{\sigma} - \frac{\omega}{\sigma\sqrt{n}}\right], \tag{1.1}$$

where $\Phi(\cdot)$ represents the standard normal distribution function.

If n is replaced by the non-negative real variable t, the random variable T is the continuous extension of the discrete variable C. The random variable T can be considered as the *time until failure* or the *failing stress level*. Thus, the distribution can now be written as

$$F(t) = P(T \le t) = \Phi\left[\frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}\right)\right], \; t > 0, \tag{1.2}$$

where $\alpha = \dfrac{\sigma}{\sqrt{\omega\mu}} > 0$ and $\beta = \dfrac{\omega}{\mu} > 0$.

Here, we describe the random variable T that has distribution function (1.2) as $T \sim$ B-S(α, β).

The distribution (1.2) has some interesting properties given in Birnbaum-Saunders (1969a). The parameter β is a scale parameter; i.e. $T/\beta \sim$ B-S$(\alpha, 1)$. Also, β is the median for the distribution since in (1.2), $F(\beta) = \Phi(0) = 0.5$. The parameter α is a shape parameter. The Birnbaum-Saunders distribution exhibits the well-known *reciprocal property*; that is, $T^{-1} \sim$ B-S(α, β^{-1}), which is in the same family of distributions (see Saunders 1974).

The probability density function (PDF) is given by

$$f(t) = F'(t) = \frac{1}{2\alpha\beta}\sqrt{\frac{\beta}{t}}\left(1 + \frac{\beta}{t}\right)\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2\alpha^2}\left(\frac{t}{\beta} - 2 + \frac{\beta}{t}\right)\right], \; t > 0, \alpha > 0, \beta > 0, \tag{1.3}$$

and is characteristically right-skewed when graphed. As α decreases, particularly for values less than unity, the density becomes nearly symmetric as the curve spread (variance) decreases. This two-parameter family of distributions has been used in various applications in reliability and life testing. For example, Birnbaum and Saunders (1969b) consider an example to model the strength of aluminum coupons subjected to cyclic stresses and strains. Three data sets are considered, each having different levels of maximum stress per cycle. The parameters in (1.3) are estimated using maximum likelihood (ML) estimation. The ML estimates for α and β must be found using numerical techniques, and the likelihood equations are given in Birnbaum and Saunders (1969b).

The derivation of the Birnbaum-Saunders distribution can be viewed as consistent with a deterministic model for material failure from mechanics known as Miner's Rule (Miner, 1945). This rule implies that for a stress level that produces a fatigue life of N cycles, the damage that occurs after n cycles is proportional to n/N; thus, (1.2) simply represents a probabilistic interpretation of this rule. When the physics of failure suggests that Miner's Rule is appropriate, the Birnbaum-Saunders model is a reasonable choice for a lifetime distribution model. However, one can imagine that in many situations in fatigue failure the assumptions made to derive (1.2) would not be completely valid. Specifically, it is assumed that the crack extension from cycle to cycle has the same distribution and that the crack growth during any one cycle is *independent* of the growth during any other cycle. (Note that this is a very different approach from the proportional degradation argument used to derive a log normal distribution (see Meeker and Escobar 1999); for that case, the rate of degradation at any point in time depends on the *total* amount of degradation that has occurred up to that time.) From a physical point of view, it might be more reasonable to assume that the crack extension at cycle k would *not* be uncorrelated with the prior k-1 cycles, but this violation of independence would invalidate the use of the classical CLT used to derive the distribution. It is for this reason that in this article we will regard the sequence of crack extensions Y_1, Y_2, …, Y_n as a stochastic process that is *not* mutually uncorrelated. The theory of long memory processes will be used to derive a new model for fatigue failure that generalizes the distribution of Birnbaum and Saunders (1969a). In the next section, we consider a new, three parameter extension of (1.2).

2. A Three Parameter Birnbaum-Saunders Distribution

Beran (1994) discusses models and statistics for samples with dependent observations. A stochastic process $Y_1, Y_2, \ldots, Y_n$ is said to have "long memory" (or long range dependence) if the *autocorrelation*

$$\rho(i, j) = \frac{E[(Y_i - \mu)(Y_j - \mu)]}{\sigma^2}$$

decays very slowly with increasing distance (either time or space) between observations. Of course, observations of this type do not satisfy the condition of independence for the distribution of the sample mean $\overline{Y}$ in the CLT; moreover, the assumptions used in classic ARMA (autoregressive moving average) or Markov-type models also do not hold. An interesting result from elementary time series is that the sample mean $\overline{Y}$ calculated from a stationary Gaussian process still exhibits a CLT-type behavior. In particular, the variance of the sample mean – although now dependent on a constant defined by the autocorrelation function – is still proportional to n^{-1}. Beran (1994) examines certain processes where the variance of $\overline{Y}$ not only differs not only by a constant multiplied by n^{-1} but also by the *speed* at which it converges to zero. He demonstrates the importance of modeling this decay explicitly when there is an indication of an alternative decay of the variance of the sample mean. Specifically, when a long memory process is self-similar with stationary increments,

$$\mathrm{Var}(\overline{Y}) = \sigma^2 n^{-2\lambda},$$

where λ is the rate of decay. If $\lambda = .5$, this is the standard result, but a slower rate of convergence would be observed with $\lambda < .5$. Moreover, if the process is Gaussian with mean μ and standard deviation σ, then $n^{-\lambda}(\overline{Y} - \mu)/\sigma$ is a standard normal variable.

Incorporating this new convergence criterion in the formulation for the Birnbaum-Saunders model (1.1) and (1.2), the distribution function becomes

$$P(C \le n) = P\left[\sum_{j=1}^{n} \frac{Y_j - \mu}{\sigma n^{\lambda}} > \frac{\omega - n^{2\lambda}\mu}{\sigma n^{\lambda}}\right] \approx \Phi\left[\frac{\mu n^{\lambda}}{\sigma} - \frac{\omega}{\sigma n^{\lambda}}\right].$$

Again by letting the random variable T represent the time until failure, the distribution can be written as

$$F(t) = \Phi\left\{\frac{1}{\alpha}\left[\left(\frac{t}{\beta}\right)^{\lambda} - \left(\frac{\beta}{t}\right)^{\lambda}\right]\right\}, t > 0, \tag{2.1}$$

where $\alpha = \dfrac{\sigma}{\sqrt{\omega\mu}} > 0$, $\beta = \left(\dfrac{\omega}{\mu}\right)^{\frac{1}{2\lambda}} > 0$, and $\lambda > 0$. This new model, a generalized three-parameter Birnbaum-Saunders distribution, includes (1.2) as a special case when $\lambda = 0.5$. The distribution (2.1) also describes, in general, the family of distributions for power transformations of T when T is distributed as (1.2). Here, we describe a random variable T that has distribution function (2.1) as T ~ GB-S(α, β, λ).

Many properties of the two parameter model (1.2) still hold for (2.1). Namely, β is a scale parameter and is the median for the distribution (2.1). In addition, this distribution retains the reciprocal property since T^{-1} ~ GB-S(α, $1/\beta$, λ).

The PDF for distribution (2.1) is given by

$$f(t) = \frac{\lambda}{\alpha\beta t\sqrt{2\pi}}\left[\left(\frac{t}{\beta}\right)^{\lambda} + \left(\frac{\beta}{t}\right)^{\lambda}\right]\exp\left\{-\frac{1}{2\alpha^2}\left[\left(\frac{\beta}{t}\right)^{2\lambda} - 2 + \left(\frac{t}{\beta}\right)^{2\lambda}\right]\right\}, t > 0. \tag{2.2}$$

Integer and fractional moments for distribution (2.2) can be expressed by noting that if T ~ GB-S(α, β, λ), then the random variable W = *ln*(T) follows a generalized case of the sinh-normal distribution (see Rieck 1999). The moment generating function for the sinh-normal distribution can be used for this purpose via the relation $M_W(r) = E[\exp(Wr)] = E(T^r)$, where r is any real number.

References

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